



TITLE:

On numerical range and norm of the generalized Aluthge transform (Role of Operator Inequalities in Operator Theory)

AUTHOR(S):

Yamazaki, Takeaki; Patel, Sharandas M.

CITATION:

Yamazaki, Takeaki ...[et al]. On numerical range and norm of the generalized Aluthge transform (Role of Operator Inequalities in Operator Theory). 数理解析研究所講究録 2005, 1427: 31-42

ISSUE DATE:

2005-04

URL:

<http://hdl.handle.net/2433/47310>

RIGHT:

On numerical range and norm of the generalized Aluthge transform

神奈川大学

山崎 文明 (Takeaki Yamazaki)

Kanagawa University

Sardar Patel University Sharandas M. Patel

ABSTRACT

In this report, we have attempted to reveal some relationships between a bounded linear operator T acting on a Hilbert space and its generalized Aluthge transformation $T(s, t)$ in terms of their numerical ranges and norms.

1. INTRODUCTION

Let $B(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} . By the polar decomposition of $T \in B(\mathcal{H})$, we mean the expression $T = U|T|$, where U is a partial isometry and $|T|$ is the positive square root of T^*T such that $\ker U = \ker |T|$. In [1], Aluthge introduced the class of p -hyponormal operators that generalizes the widely studied class of hyponormal operators. In order to reveal some important features of p -hyponormal operators, he exploited the operator \tilde{T} which is now popularly known as the *Aluthge Transformation* and which is defined as

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Motivated by this article [1], several authors explored and studied new classes of operators closely connected to p -hyponormal operators with the help of the Aluthge transformation and its generalization, known as the generalized Aluthge transformation. By the generalized Aluthge transformation of $T \in B(\mathcal{H})$, we mean the bounded operator $T(s, t)$ on \mathcal{H} for which

$$T(s, t) = |T|^s U |T|^t, \quad \text{where } s \geq 0 \text{ and } t \geq 0,$$

Especially, $T(1, 0) = |T|^1 U |T|^0 = |T| U U^* U = |T| U$ and $T(0, 1) = |T|^0 U |T|^1 = U^* U U |T|$. In recent years, one can find number of articles in which various relations among T , \tilde{T} and $T(s, t)$ are obtained. It is obvious that $\|\tilde{T}\| \leq \|T\|$. Okubo [8] gave a non-obvious extension of this inequality by deriving $\|f(\tilde{T})\| \leq \|f(T)\|$ for any polynomial $f(t)$ by proving more general result. As corollary to this inequality, he showed that $W(f(\tilde{T})) \subseteq \overline{W(f(T))}$,

extending some results known to be true [6] or in case either $f(t) = t$ [7, 9, 11]. Our main object of the present report is to compare the numerical range of T with that of $T(s, t)$ for some restricted values of s and t .

In section 2, some results are given that will be of use in the succeeding sections. Section 3 is devoted to establishing inclusion relations among the numerical ranges of rational functions of operators $T(0, 1)$, $T(1, 0)$ and T . The inequality that says $\|f(\tilde{T})\| \leq \|f(T)\|$ for every polynomial f is extended further in section 4 by proving $\|f(T(s, t))\| \leq \|f(T)\|$ with $s + t = 1$ for every rational function f for which $f(T)$ exists. Finally, in section 5, we introduce a numerical range value function on $[0, 1]$ and obtain an improvement over a characterization of convexoid matrices due to Ando [2].

In what follows, we assume, unless it is stated otherwise, that f will be a rational function with poles off $\sigma(T)$.

2. FUNDAMENTAL PROPERTIES

Lemma 2.1. *Let $T = U|T|$ be the polar decomposition of T . Then $\dim \ker T \leq \dim \ker T^*$ if and only if there exists an isometry V such that $V|T| = U|T|$.*

Although our first lemma is well known [5, p. 75], [10, p. 4], we would like to present it with a proof.

Proof. Let $\mathcal{H} = \overline{R(|T|)} \oplus R(|T|)^\perp = \overline{R(T)} \oplus R(T)^\perp$. Then U is an isometry from $\overline{R(|T|)}$ to $\overline{R(T)}$. On the other hand, there exists an isometry $U_1 : R(|T|)^\perp \rightarrow R(T)^\perp$ if and only if $\dim(R(|T|)^\perp) \leq \dim(R(T)^\perp)$. By $R(|T|)^\perp = \ker |T| = \ker T$ and $R(T)^\perp = \ker T^*$, it is equivalent to $\dim \ker T \leq \dim \ker T^*$. So the underlying kernel condition ensures the existence of an isometry $U_1 : R(|T|)^\perp \rightarrow R(T)^\perp$. Let

$$V = UU^*U + U_1(I - U^*U) = U + U_1(I - U^*U).$$

The facts that $I - U^*U$ is the projection onto $\ker U = \ker T = \ker |T| = R(|T|)^\perp = (\ker U_1)^\perp$, $U_1^*U_1x = x$ on $R(|T|)^\perp$ and $R(U_1) \subseteq R(T)^\perp = \ker T^* = \ker U^*$ will give

$$\begin{aligned} V^*V &= \{U + U_1(I - U^*U)\}^* \{U + U_1(I - U^*U)\} \\ &= U^*U + (I - U^*U)U_1^*U + U^*U_1(I - U^*U) + (I - U^*U)U_1^*U_1(I - U^*U) \\ &= U^*U + \{U^*U_1(I - U^*U)\}^* + U^*U_1(I - U^*U) + I - U^*U \\ &= U^*U + I - U^*U \\ &= I. \end{aligned}$$

Thus V is an isometry. Moreover,

$$V|T| = \{U + U_1(I - U^*U)\}|T| = U|T|.$$

□

Lemma 2.2. *Let $A \in B(\mathcal{H})$. Then the following assertions hold:*

(i) *If P is a projection with $PAP = AP$, then*

$$f(AP) = Pf(A)P + f(0)(I - P).$$

(ii) *If V is an isometry, then*

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

Proof. (i). Let $\mathcal{H} = (\ker P)^\perp \oplus \ker P$. Then by the assumption $PAP = AP$, A can be expressed as follows:

$$A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \quad \text{on } \mathcal{H} = (\ker P)^\perp \oplus \ker P.$$

Hence

$$f(AP) = f\left(\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} f(X) & 0 \\ 0 & f(0)I \end{pmatrix}.$$

On the other hand,

$$f(A) = f\left(\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}\right) = \begin{pmatrix} f(X) & Y' \\ 0 & f(Z) \end{pmatrix}.$$

Hence we have

$$Pf(A)P + f(0)(I - P) = \begin{pmatrix} f(X) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f(0)I \end{pmatrix} = f(AP).$$

(ii). For an isometry V , note that $\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix}$ is unitary. Then we have

$$\begin{aligned} \begin{pmatrix} f(VAV^*) & 0 \\ 0 & f(0)I \end{pmatrix} &= f\left(\begin{pmatrix} VAV^* & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix}\right) \\ &= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} f\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix} \\ &= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} f(A) & 0 \\ 0 & f(0)I \end{pmatrix} \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix} \\ &= \begin{pmatrix} Vf(A)V^* + f(0)(I - VV^*) & 0 \\ 0 & f(0)I \end{pmatrix}. \end{aligned}$$

Hence

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

□

The following result is a modification of [6, Proposition 4.5].

Proposition 2.3. *Let $A, B \in B(\mathcal{H})$. Then the following assertions are mutually equivalent:*

- (i) $\overline{W(f(A))} \subseteq \overline{W(f(B))}$ for all f .
- (ii) $w(f(A)) \leq w(f(B))$ for all f .
- (iii) $\|f(A)\| \leq \|f(B)\|$ for all f .

The proof is almost identical to the one given for Proposition 4.5 of [6].

3. NUMERICAL RANGES OF $T(0, 1)$ AND $T(1, 0)$

The primary object of the present section is to establish the connection among the numerical ranges of T , $T(0, 1)$ and $T(1, 0)$.

Theorem 3.1. *Let $T \in B(\mathcal{H})$. Then the following assertions hold:*

- (i) $W(f(T(0, 1))) \subseteq W(f(T))$.
- (ii) $W(f(T(1, 0))) \subseteq W(f(T))$.

Proof. (i). Let $T = U|T|$ be the polar decomposition of T , and $\mathcal{H} = (\ker T)^\perp \oplus \ker T$. Then

$$T(0, 1) = U^*UU|T| = U^*UT = U^*UTU^*U.$$

Since U^*U is a projection, (i) of Lemma 2.2 yields

$$(3.1) \quad f(T(0, 1)) = U^*Uf(T)U^*U + f(0)(I - U^*U).$$

In case $\ker T = \{0\}$. In this case U must be isometry. Then by (3.1), $f(T(0, 1)) = f(T)$, and hence

$$W(f(T(0, 1))) = W(f(T)).$$

In case $\ker T \neq \{0\}$. By (3.1), we obtain

$$W(f(T(0, 1))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\}.$$

Here by $\ker T \neq \{0\}$, we have $f(0) \in W(f(T))$, and

$$W(f(T(0, 1))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\} = W(f(T)).$$

(ii). *Step 1.* We shall show the following equality:

$$(3.2) \quad f(T(1, 0)) = U^*f(T)U + f(0)(I - U^*U).$$

We shall establish this equality separately for each of the cases when $\dim \ker T \leq \dim \ker T^*$ and $\dim \ker T \geq \dim \ker T^*$.

(a) The case $\dim \ker T \leq \dim \ker T^*$. By the Lemma 2.1, there is an isometry V satisfying $U|T| = V|T|$. Note that in the proof of Lemma 2.1,

$$(3.3) \quad V = U + U_1(I - U^*U), \text{ where } U_1 \text{ is isometry with } R(U_1) \subseteq \ker T^*.$$

Then by (3.3), we have

$$UU^*TUU^* = TUU^* = U|T|UU^* = V|T|UV^*.$$

Hence by (ii) Lemma 2.2, we obtain

$$f(TUU^*) = f(V|T|UV^*) = Vf(|T|U)V^* + f(0)(I - VV^*).$$

Moreover since V is isometry, we have

$$f(|T|U) = V^*f(TUU^*)V.$$

Therefore

$$\begin{aligned} f(|T|U) &= V^*f(TUU^*)V \\ &= V^*\{UU^*f(T)UU^* + f(0)(I - UU^*)\}V \quad \text{by (i) of Lemma 2.2} \\ &= U^*f(T)U + f(0)(I - U^*U) \quad \text{by (3.3).} \end{aligned}$$

On the other hand, by (3.3),

$$U|T^*| = UU^*UU|T|U^* = VU^*UTV^*.$$

Then by Lemma 2.2, we obtain

$$\begin{aligned} f(U|T^*|) &= f(VU^*UTV^*) \\ &= Vf(U^*UT)V^* + f(0)(I - VV^*) \quad \text{by (ii) of Lemma 2.2} \\ (3.4) \quad &= V\{U^*Uf(T)U^*U + f(0)(I - U^*U)\}V^* + f(0)(I - VV^*) \\ &\quad \text{by (i) of Lemma 2.2} \\ &= Uf(T)U^* + f(0)(I - UU^*) \quad \text{by (3.3).} \end{aligned}$$

(b) The case $\dim \ker T \geq \dim \ker T^*$. Replacing T by T^* in (3.4), we have

$$\begin{aligned} f(U^*|T|) &= U^*f(T^*)U + f(0)(I - U^*U) \\ &\iff f(|T|U) = U^*f(T)U + f(0)(I - U^*U). \end{aligned}$$

Step 2. In case $\ker T = \{0\}$. In this case U must be isometry. Then by (3.2), $f(T(1, 0)) = f(|T|U) = U^*f(T)U$, and hence

$$W(f(T(1, 0))) \subseteq W(f(T)).$$

In case $\ker T \neq \{0\}$. By (3.2), we obtain

$$W(f(T(1, 0))) = W(f(|T|U)) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\}.$$

Here by $\ker T \neq \{0\}$, we have $f(0) \in W(f(T))$, and

$$W(f(T(1, 0))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\} = W(f(T)).$$

Hence the proof is complete. □

Corollary 3.2. *Let $T = U|T|$. Then*

- (i) $W(T(1, 0)) = W(T)$ if $\ker T^* \subseteq \ker T$.
- (ii) $W(T(0, 1)) = W(T)$ if $\ker T \subseteq \ker T^*$.
- (iii) $W(T(0, 1)) \subseteq W(T(1, 0))$ if $\ker T^* \subseteq \ker T$.
- (iv) $W(T(1, 0)) \subseteq W(T(0, 1))$ if $\ker T \subseteq \ker T^*$.

Proof. (i). In view of Theorem 3.1, only we have to prove $W(T) \subseteq W(T(1, 0))$.

If $\ker T^* \subseteq \ker T$, then $T = U|T| = U|T|UU^* = UT(1, 0)U^*$, and we have

$$W(T) \subseteq \text{conv}\{W(T(1, 0)) \cup \{0\}\}.$$

If $\ker T \neq \{0\}$, then $0 \in W(T(1, 0))$ and we have

$$W(T) \subseteq \text{conv}\{W(T(1, 0)) \cup \{0\}\} = W(T(1, 0)).$$

If $\ker T = \{0\}$, then $\{0\} = \ker T \supset \ker T^*$, and U^* must be an isometry. Hence we have $W(T) \subseteq W(T(1, 0))$.

(ii). If $\ker T \subseteq \ker T^*$, then $U^*UU = U$ holds. Hence $T(0, 1) = U^*UU|T| = U|T| = T$, and $W(T(0, 1)) = W(T)$.

(iii). If $\ker T^* \subseteq \ker T$, then by (i) and Theorem 3.1, we have

$$W(T(0, 1)) \subseteq W(T) = W(T(1, 0)).$$

(iv). If $\ker T \subseteq \ker T^*$, then by (ii) and Theorem 3.1, we have

$$W(T(1, 0)) \subseteq W(T) = W(T(0, 1)).$$

□

Remark 3.3. *If we drop the kernel condition from the statements of Corollary 3.2, then we may not get the same conclusions as following indicate.*

Example 3.4. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $U = T$. Clearly $W(T(1, 0)) = \{0\} \neq W(T)$.

Example 3.5. For $\alpha > 0$, let

$$T = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$T(0, 1) = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(1, 0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Also $W(T(0,1)) = \{z \in \mathbb{C} : |z| \leq \frac{\alpha}{2}\}$ and $W(T(1,0)) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. Then

- (i) for $\alpha \in (0,1)$, $W(T(0,1)) \subsetneq W(T(1,0))$,
- (ii) for $\alpha > 1$, $W(T(1,0)) \subsetneq W(T(0,1))$.

4. NORM INEQUALITY INVOLVING A RATIONAL FUNCTION OF T AND $T(s,t)$.

Theorem 4.1. *Let $T \in B(\mathcal{H})$. Then*

$$\|f(T(s,t))\| \leq \|f(T)\|$$

holds for $s, t \geq 0$ with $s + t = 1$.

Proof. Let $T = U|T|$ be the polar decomposition of T . Let $|T_\varepsilon| = |T| + \varepsilon I > 0$. Note that

$$\lim_{\varepsilon \rightarrow +0} |T_\varepsilon|^{-1}|T| = \lim_{\varepsilon \rightarrow +0} (|T| + \varepsilon I)^{-1}|T| = U^*U.$$

We prepare the important inequality due to [4]. For $X \in B(\mathcal{H})$ and positive operators A and B ,

$$(4.1) \quad \|A^s X B^s\| \leq \|AXB\|^s \|X\|^{1-s}$$

holds for $s \in [0,1]$. Then we have

$$\begin{aligned} \|f(T(s,t))\| &= \|f(|T|^s U |T|^t)\| \\ &= \|f(|T_\varepsilon|^s |T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s |T_\varepsilon|^{-s})\| \\ &= \| |T_\varepsilon|^s f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s) |T_\varepsilon|^{-s} \| \\ &\leq \| |T_\varepsilon| f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s) |T_\varepsilon|^{-1} \|^s \|f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s)\|^t \quad \text{by (4.1)} \\ &= \|f(|T_\varepsilon|^{1-s} |T|^s U |T|^t |T_\varepsilon|^{s-1})\|^s \|f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s)\|^t \\ &\longrightarrow \|f(|T| U U^* U)\|^s \|f(U^* U U |T|)\|^t \quad \text{as } \varepsilon \rightarrow +0 \\ &= \|f(T(1,0))\|^s \|f(T(0,1))\|^t \\ &\leq \|f(T)\| \quad \text{by Theorem 3.1 and Proposition 2.3.} \end{aligned}$$

Hence the proof is complete. □

Remark 4.2. *Above theorem is not true if $s + t \neq 1$ as can be illustrated with the following Example 4.3.*

Example 4.3. *Let $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ on $\mathcal{H} = \mathbb{C}^2$. Then $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $|T| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Also $T(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It easy to find that $\overline{W(T(2,1))} = [0,2]$. Moreover $\overline{W(T)}$ is a closed elliptic disc with foci at 0 and 1, and the major axis $\sqrt{2}$ and the minor axis 1. This fact shows that $\overline{W(T)}$ excludes 2 and therefore $\overline{W(T(2,1))}$ is not a*

subset of $\overline{W(T)}$. If the theorem were true for $s+t \neq 1$, then we would have in particular, $\|T(s, t) - zI\| \leq \|T - zI\|$ for all z . Then $\overline{W(T(s, t))} \subseteq \overline{W(T)}$, which is not correct.

As simple consequence of Proposition 2.3 and Theorem 4.1, we obtain the following corollary.

Corollary 4.4. *Let $T \in B(\mathcal{H})$. Then*

$$\overline{W(f(T(s, t)))} \subseteq \overline{W(f(T))}$$

holds for $s, t \geq 0$ with $s + t = 1$.

$$5. \text{ WHEN } F(x) = \overline{W(f(T(x, 1-x)))}.$$

Theorem 5.1. *For an operator T and $x \in [0, 1]$, let*

$$F(x) = \overline{W(f(T(x, 1-x)))}.$$

Then

$$(5.1) \quad F(\alpha x + (1-\alpha)y) \subseteq \alpha F(x) + (1-\alpha)F(y)$$

holds for all $x, y \in [0, 1]$ and $\alpha \in [0, 1]$.

As a consequence of Theorem 5.1, the function $\Phi(x) = w(T(x, 1-x))$ turns out to be a convex function on $[0, 1]$.

Proof. Let $T = U|T|$ be the polar decomposition. Firstly, we shall prove

$$(5.2) \quad F\left(\frac{x+y}{2}\right) \subseteq \frac{1}{2}\{F(x) + F(y)\}.$$

Note that for a positive invertible operator S and $A \in B(\mathcal{H})$,

$$\|A\| \leq \frac{1}{2}\|SAS^{-1} + S^{-1}AS\|.$$

in [3]. Let $\varepsilon > 0$ and $|T_\varepsilon| = |T| + \varepsilon I > 0$. By the above inequality, we obtain

$$\begin{aligned} & \|f\left(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right)\right)\| \\ &= \|f(|T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}})\| \\ &\leq \frac{1}{2} \| |T_\varepsilon|^{\frac{x-y}{2}} f(|T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}}) |T_\varepsilon|^{\frac{y-x}{2}} + |T_\varepsilon|^{\frac{y-x}{2}} f(|T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}}) |T_\varepsilon|^{\frac{x-y}{2}} \| \\ &= \frac{1}{2} \|f(|T_\varepsilon|^{\frac{x-y}{2}} |T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}} |T_\varepsilon|^{\frac{y-x}{2}}) + f(|T_\varepsilon|^{\frac{y-x}{2}} |T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}} |T_\varepsilon|^{\frac{x-y}{2}})\| \\ &\longrightarrow \frac{1}{2} \|f(|T|^x U |T|^{1-x}) + f(|T|^y U |T|^{1-y})\| \quad \text{as } \varepsilon \rightarrow +0 \\ &= \frac{1}{2} \|f(T(x, 1-x)) + f(T(y, 1-y))\|. \end{aligned}$$

Hence for any complex number λ ,

$$\|f(T(\frac{x+y}{2}, 1 - \frac{x+y}{2})) - \lambda I\| \leq \|\frac{f(T(x, 1-x)) + f(T(y, 1-y))}{2} - \lambda I\|.$$

Since

$$\overline{W(T)} = \bigcap_{\lambda \in \mathbb{C}} \{z \in \mathbb{C} : |z - \lambda| \leq \|T - \lambda I\|\},$$

we have

$$\begin{aligned} F\left(\frac{x+y}{2}\right) &= \overline{W\left(f\left(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right)\right)\right)} \\ &\subseteq \overline{W\left(\frac{f(T(x, 1-x)) + f(T(y, 1-y))}{2}\right)} \\ &\subseteq \frac{1}{2}\{\overline{W(f(T(x, 1-x)))} + \overline{W(f(T(y, 1-y)))}\} \\ &= \frac{1}{2}\{F(x) + F(y)\}. \end{aligned}$$

Next, we will extend (5.2) to (5.1). From (5.2), one can easily derive

$$F\left(\frac{x_1 + x_2 + \cdots + x_{2^n}}{2^n}\right) \subseteq \frac{1}{2^n}\{F(x_1) + F(x_2) + \cdots + F(x_{2^n})\}$$

for all $x_i \in [0, 1]$ ($i = 1, 2, \dots$). Hence for any rational number $\alpha \in [0, 1]$, we have (5.1).

Since F is continuous, we have (5.1) for any real number $\alpha \in [0, 1]$.

This completes the proof. \square

Remark 5.2. The conclusion of Theorem 5.1 cannot be strengthened further to

$$F(\alpha x + (1 - \alpha)y) = \alpha F(x) + (1 - \alpha)F(y)$$

as Example 5.3 will show. However, whether the range of F is convex remains as an open problem.

Example 5.3. For $\alpha > 0$, let

$$T = \begin{pmatrix} 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$T(s, t) = \begin{pmatrix} 0 & 16^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $F(x) = W(T(x, 1-x)) = \{z : |z| \leq \frac{16^{1-x}}{2}\}$. Let $x = \frac{1}{4}$, $y = \frac{3}{4}$ and $\alpha = \frac{1}{2}$. Then

$$(i) \quad F\left(\frac{1}{2}\right) = F(\alpha x + (1 - \alpha)y) = \{z : |z| \leq 2\}.$$

$$(ii) F(x) = \{z : |z| \leq 4\}.$$

$$(iii) F(y) = \{z : |z| \leq 1\}.$$

Hence

$$\begin{aligned} F\left(\frac{1}{2}\right) &= F(\alpha x + (1 - \alpha)y) = \{z : |z| \leq 2\} \\ &\subseteq \{z : |z| \leq \frac{5}{2}\} = \alpha F(x) + (1 - \alpha) F(y). \end{aligned}$$

Corollary 5.4. *Let T be an operator. Then*

$$\begin{aligned} \overline{W(f(\tilde{T}))} &= F\left(\frac{1}{2}\right) \subseteq \frac{1}{2}\{F(s) + F(1 - s)\} \\ &\subseteq \frac{1}{2}\{F(t) + F(1 - t)\} \subseteq \overline{W(f(T))} \end{aligned}$$

holds for all $\frac{1}{2} \leq s \leq t \leq 1$.

Proof. Since $\frac{1}{2} = \frac{s+1-s}{2}$ and $F(x) = \overline{W(f(T(x, 1-x)))} \subseteq \overline{W(f(T))}$ for $x \in [0, 1]$, we have

$$\begin{aligned} \overline{W(f(\tilde{T}))} &= F\left(\frac{1}{2}\right) \subseteq \frac{1}{2}\{F(s) + F(1 - s)\} \quad \text{by Theorem 5.1} \\ &\subseteq \overline{W(f(T))} \quad \text{by Corollary 4.4.} \end{aligned}$$

Next, let $\frac{1}{2} \leq s \leq t \leq 1$. Then we have $[1 - s, s] \subseteq [1 - t, t]$. Then there exist $\alpha_1, \alpha_2 \in [0, 1]$ such that

$$s = \alpha_1 t + (1 - \alpha_1)(1 - t) \quad \text{and} \quad 1 - s = \alpha_2 t + (1 - \alpha_2)(1 - t).$$

By an easy calculation, we have $\alpha_1 + \alpha_2 = 1$, and by Theorem 5.1, we have

$$\begin{aligned} \frac{1}{2}\{F(s) + F(1 - s)\} &\subseteq \frac{1}{2}\{\alpha_1 F(t) + (1 - \alpha_1)F(1 - t) + \alpha_2 F(t) + (1 - \alpha_2)F(1 - t)\} \\ &= \frac{1}{2}\{F(t) + F(1 - t)\}. \end{aligned}$$

□

As a simple consequence of Corollary 4.4, one can see that if T is convexoid then so is $T(s, t)$ with $\overline{W(T(s, t))} = \overline{W(T)}$. The converse is obvious. However, if we do not assume $\overline{W(T(s, t))} = \overline{W(T)}$, then mere convexoidity of $T(s, t)$ does not guarantee that T is convexoid even if \mathcal{H} is finite dimensional. To see this, we refer to Example 4.3. That convexoidity of $\tilde{T} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is clear from the fact that it is selfadjoint. On the other hand as $\text{conv } \sigma(T) = [0, 1] \neq \overline{W(T)}$, T is not convexoid. However, if \mathcal{H} is finite-dimensional, then our next result will show that the condition $\overline{W(T(s, t))} = \overline{W(T)}$ is just equivalent to the convexoidity of T . In case \mathcal{H} is infinite dimensional, we do not know the validity of this result.

Corollary 5.5. *For a $n \times n$ matrix T , the following assertions are mutually equivalent:*

- (i) T is convexoid.
- (ii) $W(\tilde{T}) = W(T)$.
- (iii) $W(T(s_0, 1 - s_0)) = W(T)$ for a fixed $s_0 \in (0, 1)$.
- (iv) $W(T(s, 1 - s)) = W(T)$ for all $s \in [0, 1]$.

In order to prove Corollary 5.5, we shall need the following theorem, a remarkable result due to Ando [2].

Theorem A ([2]). *Let T be a $n \times n$ matrix. Then T is convexoid if and only if $W(\tilde{T}) = W(T)$.*

Proof. (i) \iff (ii) has been shown in Theorem A. (iv) \implies (ii), (iii) are obvious. So only we have to show (ii) \implies (iv) and (iii) \implies (ii).

Proof of (ii) \implies (iv). Since $W(T(s, 1 - s)) \subseteq W(T)$ and $W(T(1 - s, s)) \subseteq W(T)$ for all $s \in [0, 1]$ hold and Corollary 5.4, we have

$$\begin{aligned} W(T) = W(\tilde{T}) &\subseteq \frac{1}{2} \{W(T(s, 1 - s)) + W(T(1 - s, s))\} \\ &\subseteq \frac{1}{2} \{W(T(s, 1 - s)) + W(T)\} \subseteq W(T). \end{aligned}$$

Then we have

$$(5.3) \quad \frac{1}{2} \{W(T(s, 1 - s)) + W(T)\} = W(T).$$

For any $\theta \in [0, 2\pi)$, let λ be an extreme point of $\operatorname{Re} e^{i\theta} W(T)$. Then by (5.3), there exist $\lambda_1 \in \operatorname{Re} e^{i\theta} W(T)$ and $\mu_1 \in \operatorname{Re} e^{i\theta} W(T(s, 1 - s))$ such that

$$\lambda = \frac{\lambda_1 + \mu_1}{2}.$$

Since $\operatorname{Re} e^{i\theta} W(T)$ is a line segment, and λ is a extreme point of $\operatorname{Re} e^{i\theta} W(T)$, it must be $\lambda = \lambda_1 = \mu_1 \in \operatorname{Re} e^{i\theta} W(T(s, 1 - s))$, i.e., $\operatorname{Re} e^{i\theta} W(T) \subseteq \operatorname{Re} e^{i\theta} W(T(s, 1 - s))$ for any $\theta \in [0, 2\pi)$. Since $W(T)$ is convex, and $W(T(s, 1 - s)) \subseteq W(T)$ always holds, we have $W(T) = W(T(s, 1 - s))$ for all $s \in [0, 1]$.

Proof of (iii) \implies (ii). We may assume $s_0 > \frac{1}{2}$. For each $s_0 \in (\frac{1}{2}, 1)$, there exists $\alpha \in (0, 1)$ such that

$$s_0 = \alpha \frac{1}{2} + (1 - \alpha) \cdot 1.$$

Then by Theorem 5.1,

$$W(T) = W(T(s_0, 1 - s_0)) \subseteq \alpha W(\tilde{T}) + (1 - \alpha) W(T(1, 0)) \subseteq W(T).$$

By the same argument of the above one, we have $W(\tilde{T}) = W(T)$. □

Remark 5.6. In (iii) of Corollary 5.5, s_0 must not be 0 or 1, because if T is invertible, then U is unitary and $W(T) = W(T(0, 1)) = W(T(1, 0))$. But in general, $W(T) \neq W(\tilde{T})$.

REFERENCES

- [1] A. Aluthge, *On p -hyponormal operators for $0 < p < 1$* , Integral Equations Operator Theory, **13** (1990), 307–315.
- [2] T. Ando, *Aluthge transforms and convex hull of eigenvalues of a matrix*, Linear and Multilinear Algebra, **52** (2004), 281–292.
- [3] G. Corach, H. Porta and L. Recht, *An operator inequality*, Linear Algebra Appl., **142** (1990), 153–158.
- [4] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438.
- [5] P. R. Halmos, *A Hilbert Space Problem Book 2nd ed.*, Springer Verlag, New York, 1982.
- [6] M. Ito, H. Nakazato, K. Okubo and T. Yamazaki, *On generalized numerical range of the Aluthge transformation*, Linear Algebra Appl., **370** (2003), 147–161.
- [7] I. B. Jung, E. Ko and C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory, **37** (2000), 437–448.
- [8] K. Okubo, *On weakly unitarily invariant norm and the Aluthge transformation*, Linear Algebra Appl., **371** (2003), 269–375.
- [9] P. Y. Wu, *Numerical range of Aluthge transform of operator*, Linear Algebra Appl., **357** (2002), 295–298.
- [10] D. Xia, *Spectral theory of hyponormal operators*, Birkhäuser Verlag, Basel, 1983.
- [11] T. Yamazaki, *On numerical range of the Aluthge transformation*, Linear Algebra Appl., **341** (2002), 111–117.